Method of Separation of Variables Applied to the BVP (14).

\[ \psi(r, \theta) = \phi(\theta) G(r) \]  

(15.1)

\[ \nabla_{re}^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0 \]  

(15.2)

Substituting (15.1) into (15.2)

\[ \phi(\theta) \frac{1}{r} \frac{d}{dr} \left( r \frac{dG}{dr} \right) + \frac{1}{r^2} G(r) \frac{d^2 \phi}{d\theta^2} = 0 \]

Multiplying by \( \frac{r^2}{\phi(\theta) G(r)} \)

\[ \frac{r}{G(r)} \frac{d}{dr} \left( r \frac{dG}{dr} \right) + \frac{1}{\phi(\theta)} \frac{d^2 \phi}{d\theta^2} = 0 \]

or \[ \frac{r}{G(r)} \frac{d}{dr} \left( r \frac{dG}{dr} \right) = -\frac{1}{\phi(\theta)} \frac{d^2 \phi}{d\theta^2} = \lambda \]
Thus, \[
\frac{d^2 \phi}{d\theta^2} + \lambda \phi = 0 \quad (16.1)
\]
and \[
r \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) - \lambda G = 0 \quad (16.2)
\]

\(\phi(\theta)\) also satisfies periodic B.C.'s:
\[
\phi(-\pi) = \phi(\pi) \quad \text{and} \quad \frac{d\phi}{d\theta} (-\pi) = \frac{d\phi}{d\theta} (\pi)
\]

What is the meaning of these conditions?

Therefore, \(\phi(\theta)\) satisfies the eigenvalue problem:

\[
\begin{cases}
\frac{d^2 \phi}{d\theta^2} + \lambda \phi = 0, & -\pi < \theta < \pi, \\
\phi(-\pi) = \phi(\pi), & \phi'(-\pi) = \phi'(\pi)
\end{cases}
\]

This is a well known eigenvalue problem whose eigenvalues and eigenfunctions are (See Helmer)

\[
\lambda_n = n^2, \quad n = 0, 1, 2, \ldots
\]

and \(\phi_n(\theta) = \left\{ \begin{array}{ll}
\sin(n\theta) & n = 0, 1, 2, \ldots \\
\cos(n\theta) & n = 0
\end{array} \right.\)
The equations for $G(r)$ are now given by

$$r \frac{d}{dr} \left( r \frac{dg}{dr} \right) - n^2 G = 0, \quad n=0,1,2, \ldots$$

(17.1)

If $n=0$

$$r \frac{d}{dr} \left( r \frac{dg}{dr} \right) = 0$$

$$\Rightarrow \frac{dg}{dr} = \frac{C_1}{r}$$

$$\Rightarrow G(r) = C_2 \ln(r) + C_2$$

(17.2)

If $n \neq 0$

Equ. (17.1) may be written as

$$r^2 \frac{d^2 G}{dr^2} + r \frac{dg}{dr} - n^2 G = 0$$

which is Euler's equation. Look for solutions in the form: $G(r) = r^p$, and find out $p = \pm n$

Since $G_1(r) = r^n$ and $G_2(r) = r^{-n}$ are linearly independent solutions of (17.1) for $n \neq 0$, the general solution can be written as

$$G(r) = C_1 r^n + C_2 r^{-n}$$

(17.3)
Applying the principle of superposition, we obtain
as solution of Laplace's equation for \( \psi(r, \theta) \)

\[
\psi(r, \theta) = C_1 \ln r + C_2 + \sum_{n=1}^{\infty} \left[ A_n r^n + B_n r^{-n} \right] \sin(\theta) + \\
+ \sum_{n=1}^{\infty} \left[ D_n r^n + E_n r^{-n} \right] \cos(\theta). \quad (18.1)
\]

Determining constants from B.C.'s:

B.C. (14.3): \( \psi(r, \theta) \xrightarrow{r \to \infty} y = r \sin \theta \)

It's equivalent to

\[
C_1 \ln r + C_2 + A_1 r \sin \theta + B_1 \frac{1}{r} \sin \theta + A_2 r^2 \sin(2\theta) + B_2 \frac{1}{r^2} \sin(2\theta) + \ldots \\
+ D_1 r \cos \theta + E_1 \frac{1}{r} \cos \theta + D_2 r^2 \cos(2\theta) + E_2 \frac{1}{r^2} \cos(2\theta) + \ldots
\]

\( r \to \infty \)

\( y \sin \theta \)
Therefore,

\[ A_i = 1, \]

\[ B_n \ (n \geq 1); \] they are all undetermined, since \[ \lim_{r \to \infty} \frac{1}{r^n} \sin(n\theta) = 0 \]

\[ D_1 = 0; \] because the dominant and only term in the right-hand side is rsin\theta.

The term \[ D_1 \cos \theta \] is of the same \( O(r) \), but it would not have a counterpart in the rhs.

\[ D_n = 0 \ (n \geq 2); \] because they are \( O(r^n) \) and they would dominate the term rsin\theta in the rhs.

\[ E_n \ (n \geq 1): \] they are undetermined because all terms \[ \lim_{r \to \infty} \frac{1}{r^n} \cos(n\theta) = 0 \]

Finally, the terms \( C_1/r \sin \theta + C_2 \) are dominated by rsin\theta. More precisely,

\[ C_1/r \sin \theta + C_2 + r \sin \theta \quad \xrightarrow{r \to \infty} \quad r \sin \theta. \]

Therefore, \( C_1 \) and \( C_2 \) remain undetermined.
Thus, the solution (18.1) reduces to

\[ \psi(r, \theta) = C_1 \ln r + C_2 + \left( r + \beta_1 \frac{1}{r} \right) \sin \theta + \]

\[ + \sum_{n=2}^{\infty} \beta_n r^{-n} \sin (n\theta) + \sum_{n=1}^{\infty} \alpha_n r^{-n} \cos (n\theta). \]

Applying B.C. (14.2): \( \psi(1, \theta) = 0 \)

we obtain

\[ C_1 \ln 1 + C_2 + \left( 1 + \beta_1 \right) \sin \theta + \beta_2 \sin(2\theta) + \ldots \]

\[ + \alpha_1 \cos \theta + \alpha_2 \cos(2\theta) + \ldots = 0 \]

Then

\[ C_2 = 0, \quad \beta_1 = -1, \quad \beta_n = 0, \quad n > 2; \]

\[ \alpha_n = 0, \quad n > 1. \]

The final answer is given by

\[ \psi(r, \theta) = C_1 \ln r + \left( r - \frac{1}{r} \right) \sin \theta \quad (20.1) \]

Therefore, there are infinitely many solutions depending on \( C_1. \)