Integral Equations.

Unknown function is under integral sign.

Examples:

Fredholm equation:

\[ \int_a^b K(x,y) u(y) \, dy + \lambda(x) u(x) = f(x), \quad a \leq x \leq b. \quad (1) \]

Volterra equation:

\[ \int_a^x K(x,y) u(y) \, dy + \lambda(x) u(x) = f(x), \quad a \leq x \leq b. \quad (2) \]

\( u(x,y) \): unknown function.

\( K(x,y) \): kernel

If \( f(x) \equiv 0 \) equations are called homogeneous.

Otherwise nonhomogeneous.

If \( \lambda(x) \equiv 0 \) equations are said to be of the first kind, otherwise they are of 2nd kind.
Integral Operator:

\[ K[u](x) = \int_a^b k(x,y) u(y) \, dy. \]

Then (1) can be written as

\[ K[u] + \lambda u = f \]

Eigenvalues: Any \( \mu \) such that there exists \( u(x) \neq 0 \) and \( K[u] = \mu u \)

Spectrum = \{ Set of eigenvalues \}.

Inner Product and Norm: \( u, v \in C[a,b] \) (or more general \( u, v \in L^2[a,b] \))

Inner Product:

\[ (u,v) = \int_a^b u(x) \overline{v(x)} \, dx. \]

Norm:

\[ \|u\| = \sqrt{\int_a^b |u(x)|^2 \, dx}. \]
Example of a Volterra Equation:

Inventory control Problem:

$A$: Amount of goods initially purchased ($t=0$).

$U_k(t)$: Rate at which goods are purchased.

$K(t)$: Percentage of goods that remains unsold at time $t$, after purchasing the goods.

Condition: Want stocks to remain constant.

Math Model:

At what rate the goods need to be purchased for the stock to remain constant?

$$AK(t) + \int_0^t K(t-r) U(r) \, dr = A.$$ 

Why? Consider the interval $[t, t+\Delta t]$:

a) Amount purchased in time interval $\Delta t$:

$$U(\bar{t}) \Delta t,$$

$\bar{t} \in [t, t+\Delta t]$.

b) Portion remaining unsold at time $t$:

$$K(t-\bar{t}) U(\bar{t}) \Delta t,$$
If \( \Delta r \to 0 \), we get the portion of \( \Delta r \) remaining unsold at \( t = T \). Therefore, adding \( \int_0^t u(r) \, dr \) we will get the total portion remaining unsold at time \( t \).

\[
\int_0^t k(t-r)u(r) \, dr
\]

This is not complete until we add the portion of the initial amount \( a \) that remains unsold at \( t \).

\[
A k(t) + \int_0^t k(t-r)u(r) \, dr = A.
\]
Laplace Transform applied to a special Volterra I. Eqn.

Consider

$$U(x) = f(x) + \int_{0}^{x} k(x-y) u(y) dy.$$  \hspace{1cm} (5.1)

Integral equation of convolution type.

$$\int_{0}^{x} k(x-y) u(y) dy = k \ast u.$$  

We know

Laplace transform.

$$L\{k \ast u\} = L\{k\} L\{u\}.$$  

Taking L.T. of (5.1).

$$L\{U\} = L\{f\} + L\{k\} L\{u\}.$$  

$$\Rightarrow \quad L\{u\} = \frac{L\{f\}}{1 - L\{k\}}.$$  

In particular, if we want to solve

\[ U(x) = x - \int_0^x \frac{K(x-y)}{(x-y)} u(y) \, dy \]

\[ \Rightarrow K(x) = x. \]

Applying L.T.

\[ \mathcal{L}\{u(t)\} = \mathcal{L}\{x\} - \mathcal{L}\{u(0)\} \times 1 \]

\[ = \frac{1}{s^2} - \frac{1}{s^2} \mathcal{L}\{u(0)\} \]

\[ \Rightarrow \mathcal{L}\{u(t)\} = \frac{\frac{1}{s^2}}{1 + \frac{1}{s^2}} = \frac{1}{s^2+1} \]

\[ \Rightarrow \mathcal{L}^{-1}\{u(t)\} = e^{-t} \sin t \]

Reformulation of IVP as a Volterra Eqns.

\[ U'(x) = f(x, u), \quad u(x_0) = u_0 \]

\[ \int_{x_0}^x u'(y) \, dy = \int_{x_0}^x f(y, u(y)) \, dy \]

\[ \text{FTC.} \]

\[ U(x) = U(x_0) + \int_{x_0}^x f(y, u(y)) \, dy \]
Second order IVP (ODE) $\rightarrow$ Volterra Integ. Eqn.

Lemma: $f(x)$ continuous for $x \geq a$. Then

$$\int_a^x \left( \int_a^s f(y) \, dy \right) \, ds = \int_a^x f(y) \, (x-y) \, dy$$

Proof:

$$\int_a^x \left( \int_a^s f(y) \, dy \right) \, ds = \int_a^x F(s) \, ds = F(x) - F(a) - \int_a^x F'(s) \, ds$$

$$= x \int_a^x f(y) \, dy - \int_a^x s \, f(s) \, ds = x \int_a^x f(y) \, dy - \int_a^x y f(y) \, dy$$

$$\therefore \int_a^x \int_a^s f(y) \, dy \, ds = \int_a^x f(y) \, (x-y) \, dy$$
Example: Consider

\[
\begin{cases}
U'' + p(x)U' + q(x)U = f(x), & x > a \\
U(a) = u_0, & U'(a) = u_1
\end{cases}
\]

\[U'' = -p(x)U' - q(x)U + f(x).\]

\[\int_{a}^{x} dy \quad U'(x) - U'(a) = - \int_{a}^{x} p(y)U'(y) \, dy - \int_{a}^{x} [q(y)U(y) - f(y)] \, dy\]

\[\Rightarrow \quad U'(x) - U'(a) = -p(x)U(x) + p(a)U(a) - \int_{a}^{x} p(y)U'(y) \, dy - \int_{a}^{x} [q(y)U(y) - f(y)] \, dy\]

\[\therefore \quad U'(x) - U_1 = -p(x)U(x) + p(a)U_0 - \int_{a}^{x} [q(y)U(y) - f(y)] \, dy\]

\[\int_{a}^{x} dy \quad U(x) - U(a) - U_1(x-a) = -\int_{a}^{x} p(x)U(x) \, dx + p(a)U_0(x-a) - \int_{a}^{x} \left( \int_{a}^{s} \left\{ q(y) - p'(y) \right\} U(y) - f(y) \right) \, dy \, ds\]
Using lemma.

\[ U(x) = U_0 + \left( p(a) + U_1 \right) (x-a) - \int_a^x p(x) U(x) \, dx \]

\[ - \int_a^x \left( q(y) - p'(y) \right) U(y) (x-y) \, dy \]

\[ + \int_a^x f(y) (x-y) \, dy \]

or

\[ U(x) = U_0 + \left[ p(a) + U_1 \right] (x-a) - \int_a^x \left[ p(y) + \left[ q(y) - p'(y) \right] (x-y) \right] U(y) \, dy \]

\[ + \int_a^x f(y) (x-y) \, dy . \]

This equation is of the form of a Volterra equ. of the second kind.

\[ U(x) = \int_a^x K(x,y) U(y) \, dy + F(x) . \]