

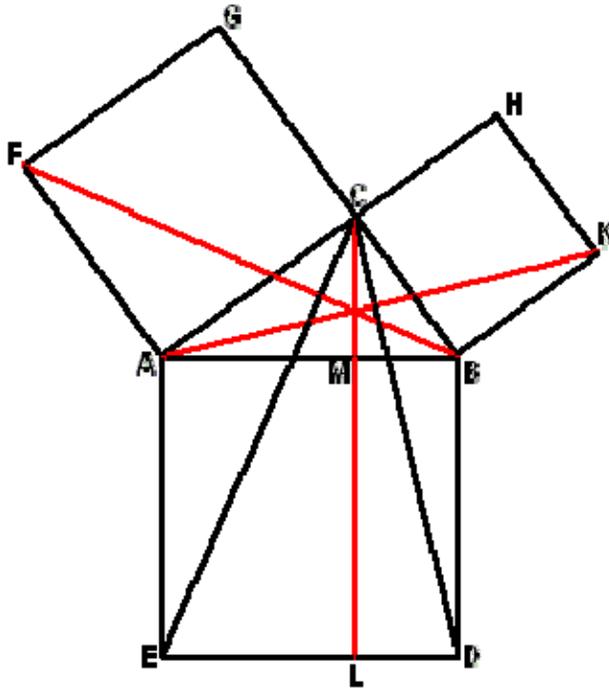
Pythagorean Theorem

Facts:

1. The statement of the Theorem was discovered on a Babylonian tablet circa 1900-1600 B.C. Whether Pythagoras (c. 560-c.480 B.C.) or someone else from his School was the first to discover its proof can't be claimed with any degree of credibility. Euclid's (c 300 B.C.) Elements furnish the first and, later, the standard reference in Geometry. In fact Euclid supplied two very different proofs. The Theorem is reversible which means that a triangle whose sides satisfy the equality is necessarily right angled. Euclid was the first to mention and prove this fact.
2. The Pythagorean Theorem generalizes to spaces of higher dimensions. Some of the generalizations are far from obvious.
3. Wherever all three sides of a right triangle are integers, their lengths form a Pythagorean triple (or Pythagorean numbers). There is a general formula for obtaining all such numbers. (Let n and m be integers with $n > m$. Let $a = n^2 - m^2$, $b = 2nm$, $c = n^2 + m^2$. Then a, b, c form a Pythagorean triple.)
4. The Pythagorean Theorem is equivalent to the Parallel Postulate.

Proof #1

This is probably the most famous of all proofs of the Pythagorean proposition. It's the first of Euclid's two proofs. The underlying configuration became known under a variety of names, the [*Bride's Chair*](#) likely being the most popular.



The proof has been illustrated by an award winning Java applet written by Jim Morey. You can see it at <http://www.cut-the-knot.org/pythagoras/morey.shtml>. The proof below is a somewhat shortened version of the original Euclidean proof as it appears in Sir Thomas Heath's translation.

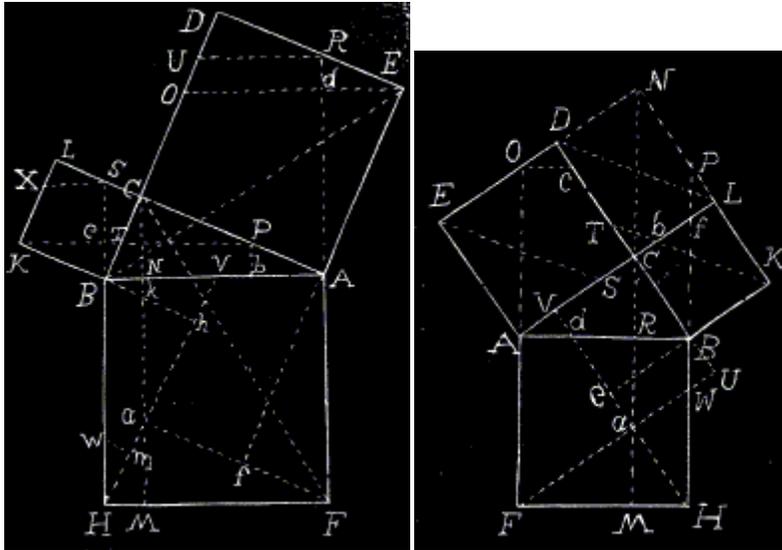
First of all, $\triangle ABF = \triangle AEC$ by SAS. This is because, $AE = AB$, $AF = AC$, and

$$\angle BAF = \angle BAC + \angle CAF = \angle CAB + \angle BAE = \angle CAE.$$

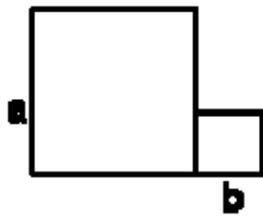
$\triangle ABF$ has base AF and the altitude from B equal to AC . Its area therefore equals half that of square on the side AC . On the other hand, $\triangle AEC$ has AE and the altitude from C equal to AM , where M is the point of intersection of AB with the line CL parallel to AE . Thus the area of $\triangle AEC$ equals half that of the rectangle $AELM$. Which says that the area AC^2 of the square on side AC equals the area of the rectangle $AELM$.

Similarly, the area BC^2 of the square on side BC equals that of rectangle $BMLD$. Finally, the two rectangles $AELM$ and $BMLD$ make up the square on the hypotenuse AB .

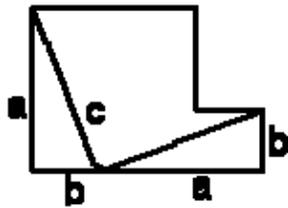
The configuration at hand admits numerous variations. B. F. Yanney and J. A. Calderhead (*Am Math Monthly*, v.4, n 6/7, (1987), 168-170) published several proofs based on the following diagrams



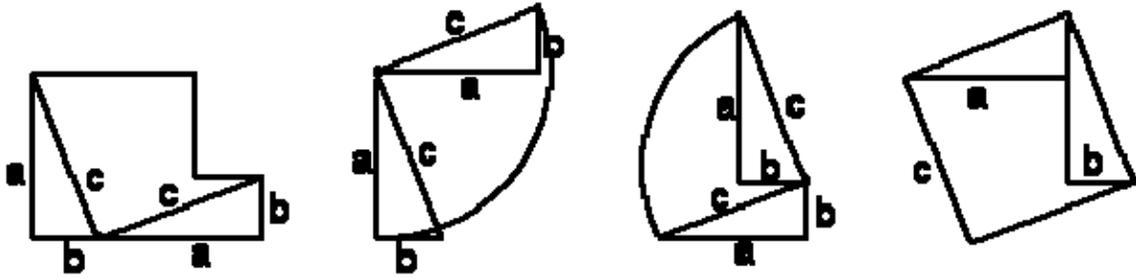
Proof #2



We start with two squares with sides **a** and **b**, respectively, placed side by side. The total area of the two squares is a^2+b^2 .



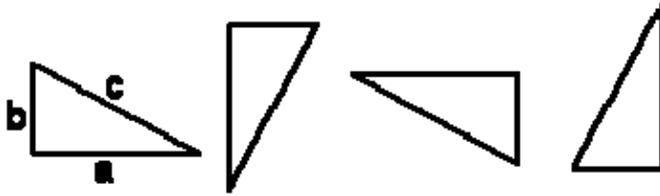
The construction did not start with a triangle but now we draw two of them, both with sides **a** and **b** and hypotenuse **c**. Note that the segment common to the two squares has been removed. At this point we therefore have two triangles and a strange looking shape.



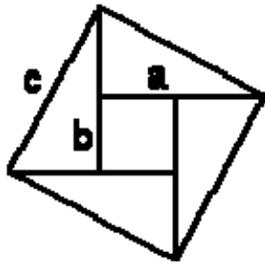
As a last step, we rotate the triangles 90° , each around its top vertex. The right one is rotated clockwise whereas the left triangle is rotated counterclockwise. Obviously the resulting shape is a square with the side c and area c^2 . This proof appears in a dynamic incarnation at <http://www.cut-the-knot.org/Curriculum/Geometry/HingedPythagoras3.shtml>.



Proof #3



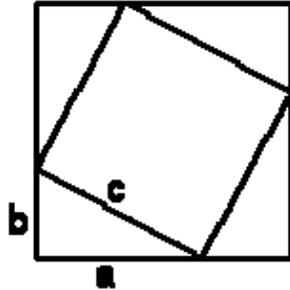
Now we start with four copies of the same triangle. Three of these have been rotated 90° , 180° , and 270° , respectively. Each has area $ab/2$. Let's put them together without additional rotations so that they form a square with side c .



The square has a square hole with the side $(a - b)$. Summing up its area $(a - b)^2$ and $2ab$, the area of the four triangles ($4 \cdot ab/2$), we get

$$\begin{aligned}
 c^2 &= (a - b)^2 + 2ab \\
 &= a^2 - 2ab + b^2 + 2ab \\
 &= a^2 + b^2
 \end{aligned}$$

Proof #4

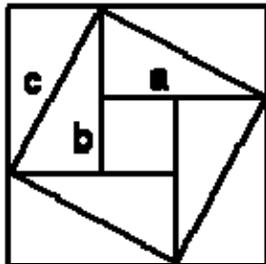


The fourth approach starts with the same four triangles, except that, this time, they combine to form a square with the side $(a + b)$ and a hole with the side c . We can compute the area of the big square in two ways. Thus

$$(a + b)^2 = 4 \cdot ab/2 + c^2$$

simplifying which we get the needed identity.

A proof which combines this with [proof #3](#) is credited to the 12th century Hindu mathematician Bhaskara (Bhaskara II):



Here we add the two identities

$$\begin{aligned} c^2 &= (a - b)^2 + 4 \cdot ab/2 \text{ and} \\ c^2 &= (a + b)^2 - 4 \cdot ab/2 \end{aligned}$$

which gives

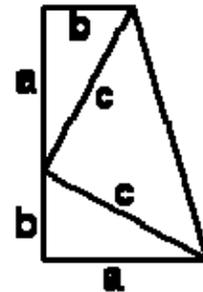
$$2c^2 = 2a^2 + 2b^2.$$

The latter needs only be divided by 2.



Proof #5

This proof, discovered by President J.A. Garfield in 1876, is a variation on the previous one. But this time we draw no squares at all. The key now is the formula for the area of a trapezoid - *half sum of the bases times the altitude* - $(a + b)/2 \cdot (a + b)$. Looking at the picture another way, this also can be computed as the sum of areas of the three triangles - $ab/2 + ab/2 + c \cdot c/2$. As before, simplifications yield $a^2 + b^2 = c^2$.



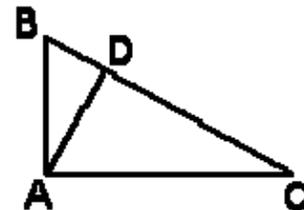
Two copies of the same trapezoid can be combined in two ways by attaching them along the slanted side of the trapezoid. One leads to the [proof #4](#), the other to [proof #52](#).



Proof #6

We start with the original triangle, now denoted ABC, and need only one additional construct - the altitude AD. The triangles ABC, BDA and ADC are similar which leads to two ratios:

$$AB/BC = BD/AB \text{ and } AC/BC = DC/AC.$$



Written another way these become

$$AB \cdot AB = BD \cdot BC \text{ and } AC \cdot AC = DC \cdot BC$$

Summing up we get

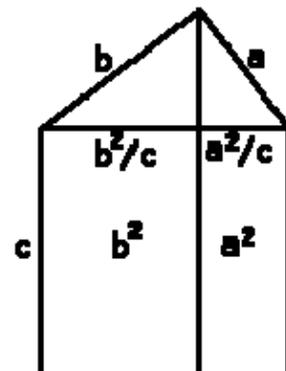
$$\begin{aligned} AB \cdot AB + AC \cdot AC &= BD \cdot BC + DC \cdot BC \\ &= (BD+DC) \cdot BC = BC \cdot BC. \end{aligned}$$

In a little different form, this proof appeared in the *Mathematics Magazine*, 33 (March, 1950), p. 210, in the Mathematical Quickies section.

Taking $AB = a$, $AC = b$, $BC = c$ and denoting $BD = x$, we obtain as above

$$a^2 = cx \text{ and } b^2 = c(c - x),$$

which perhaps more transparently leads to the same identity.



In a private correspondence, Dr. France Dacar, Ljubljana, Slovenia, has suggested that the diagram on the right may serve two purposes. First, it gives an additional graphical representation to the present proof #6. In addition, it highlights the relation of the latter to proof #1.

R. M. Mentock has observed that a little trick makes the proof more succinct. In the common notations, $c = b \cos A + a \cos B$. But, from the original triangle, it's easy to see that $\cos A = b/c$ and $\cos B = a/c$ so $c = b (b/c) + a (a/c)$. This variant immediately brings up a question: are we getting in this manner a trigonometric proof? I do not think so, although a trigonometric function (cosine) makes here a prominent appearance. The ratio of two lengths in a figure is a shape property meaning that it remains fixed in passing between similar figures, i.e., figures of the same shape. That a particular ratio used in the proof happened to play a sufficiently important role in trigonometry and, more generally, in mathematics, so as to deserve a special notation of its own, does not cause the proof depend on that notation.

Finally, it must be mentioned that the configuration exploited in this proof is just a specific case of the one from the next proof - Euclid's second and less known proof of the Pythagorean proposition.



Proof #7

The next proof is taken verbatim from Euclid VI.31 in translation by Sir Thomas L. Heath. The great G. Polya analyzes it in his *Induction and Analogy in Mathematics* (II.5) which is a recommended reading to students and teachers of Mathematics.

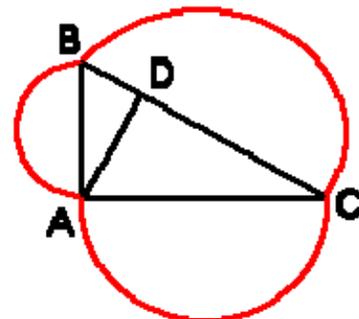
In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

Let ABC be a right-angled triangle having the angle BAC right; I say that the figure on BC is equal to the similar and similarly described figures on BA, AC.

Let AD be drawn perpendicular. Then since, in the right-angled triangle ABC, AD has been drawn from the right angle at A perpendicular to the base BC, the triangles ABD, ADC adjoining the perpendicular are similar both to the whole ABC and to one another [VI.8].

And, since ABC is similar to ABD, therefore, as CB is to BA so is AB to BD [VI.Def.1].

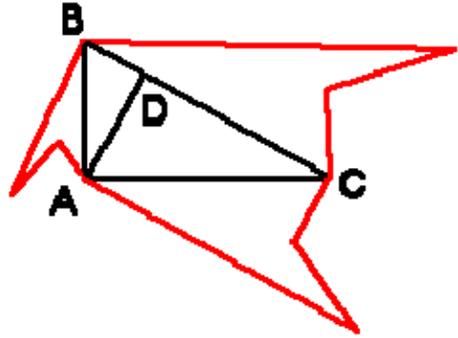
And, since three straight lines are proportional, as the first is to the third, so is the figure on the first to the similar and similarly described figure on the second [VI.19]. Therefore, as CB is to BD, so is the figure on CB to the similar and similarly described figure on BA.



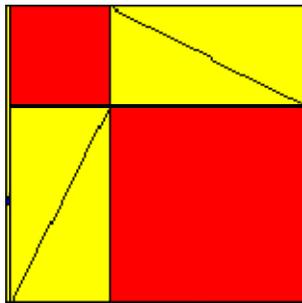
For the same reason also, as BC is to CD, so is the figure on BC to that on CA; so that, in addition, as BC is to BD, DC, so is the figure on BC to the similar and similarly described figures on BA, AC.

But BC is equal to BD, DC; therefore the figure on BC is also equal to the similar and similarly described figures on BA, AC.

Therefore etc. Q.E.D.

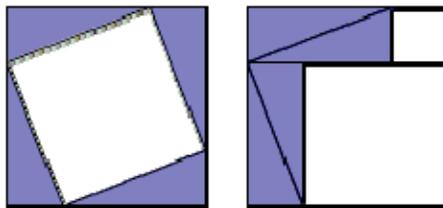


Proof #9



Another proof stems from a rearrangement of rigid pieces, much like [proof #2](#). It makes the algebraic part of [proof #4](#) completely redundant. There is nothing much one can add to the two pictures.

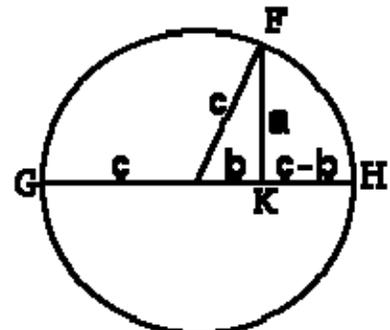
(The proof has been published by Rufus Isaac in *Mathematics Magazine*, Vol. 48 (1975), p. 198.)



A slightly different rearrangement leads to a hinged dissection illustrated by a Java applet found at <http://www.cut-the-knot.org/Curriculum/Geometry/HingedPythagoras.shtml>.



Proof #11



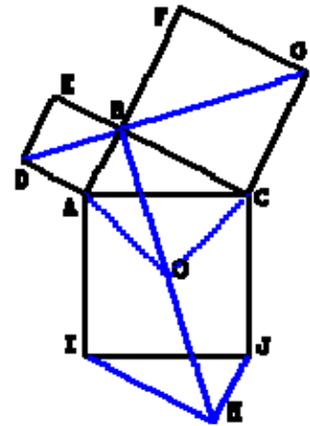
Draw a circle with radius c and a right triangle with sides a and b as shown. In this situation, one may apply any of a few well known facts. For example, in the diagram three points F, G, H located on the circle form another right triangle with the altitude FK of length a . Its hypotenuse GH is split in two pieces: $(c + b)$ and $(c - b)$. So, as in [Proof #6](#), we get $a^2 = (c + b)(c - b) = c^2 - b^2$.

B. F. Yanney and J. A. Calderhead (*Am Math Monthly*, v.3, n. 12 (1896), 299-300) offer a somewhat different route. Imagine FK is extended to the second intersection F' with the circle. Then, by the [Intersecting Chords](#) theorem, $FK \cdot KF' = GK \cdot KH$, with the same implication.

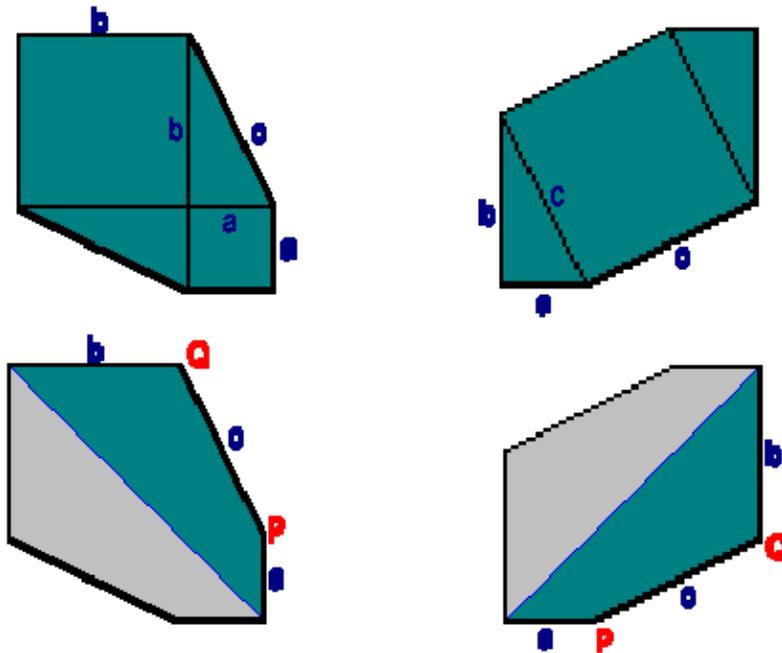


Proof #16

This proof is ascribed to Leonardo da Vinci (1452-1519) [[Eves](#)]. Quadrilaterals $ABHI$, $JHBC$, $ADGC$, and $EDGF$ are all equal. (This follows from the observation that the angle ABH is 45° . This is so because ABC is right-angled, thus center O of the square $ACJI$ lies on the circle circumscribing triangle ABC . Obviously, angle ABO is 45° .) Now, $\text{Area}(ABHI) + \text{Area}(JHBC) = \text{Area}(ADGC) + \text{Area}(EDGF)$. Each sum contains two areas of triangles equal to ABC (IJH or BEF) removing which one obtains the Pythagorean Theorem.



David King modifies the argument somewhat



The side lengths of the hexagons are identical. The angles at P (right angle + angle between a & c) are identical. The angles at Q (right angle + angle between b & c) are identical. Therefore all four hexagons are identical.

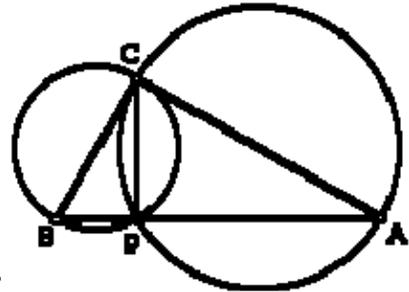


Proof #22

Here is the second proof from Dr. Scott Brodie's letter.

We take as known a "power of the point" theorems: If a point is taken exterior to a circle, and from the point a segment is drawn tangent to the circle and another segment (a secant) is drawn which cuts the circle in two distinct points, then the square of the length of the tangent is equal to the product of the distance along the secant from the external point to the nearer point of intersection with the circle and the distance along the secant to the farther point of intersection with the circle.

Let ABC be a right triangle, with the right angle at C. Draw the altitude from C to the hypotenuse; let P denote the foot of this altitude. Then since CPB is right, the point P lies on the circle with diameter BC; and since CPA is right, the point P lies on the circle with diameter AC. Therefore the intersection of the two circles on the legs BC, CA of the original right triangle coincides with P, and in particular, lies on AB. Denote by x and y the lengths of segments BP and PA, respectively, and, as usual let a , b , c denote the lengths of the sides of ABC opposite the angles A, B, C respectively. Then, $x + y = c$.



Since angle C is right, BC is tangent to the circle with diameter CA, and the power theorem states that $a^2 = xc$; similarly, AC is tangent to the circle with diameter BC, and $b^2 = yc$. Adding, we find $a^2 + b^2 = xc + yc = c^2$, Q.E.D.



See <http://www.cut-the-knot.org/pythagoras/index.shtml> for more information.