Definition: A number $a$ is divisible by $c$ if there is a whole number $n$ such that $a = nc$. We denote it by $c|a$.

Examples: $15$ is divisible by $3$, i.e., $15 = 5 \times 3$.

Properties of Divisibility:

Property 1: If $a$ and $b$ are divisible by $c$, i.e., $c|a$ and $c|b$, then $c|(a + b)$ and $c|(a - b)$.

Example: $2|10$ and $2|6$ imply that $2|(10 + 6)$.

Property 2: If $bc|a$, then $b|a$ and $c|a$.

Example: $(2 \times 5)|40$ implies $2|40$ and $5|40$.

Property 3: If $b|a$ and $c|a$, and $(b, c) = 1$ (Namely, the largest common factor of $b$ and $c$ is $1$), then $bc|a$.

Example: $2|28$ and $7|28$ and $(2, 7) = 1$ imply that $(2 \times 7)|28$.

Features of Divisibility:

1. A number is divisible by $2$. The last digit of the number is either $0, 2, 4, 6,$ or $8$.
2. A number is divisible by $5$. The last digit of the number is either $0$ or $5$.
3. A number is divisible by $3$ or $9$. The summation of all digits of the number is divisible by $3$ or $9$.
4. A number is divisible by $4$ or $25$. The last two digits is divisible by $4$ or $25$.
5. A number is divisible by $8$. The last three digits of the number is divisible by $8$.
6. A number is divisible by $7$ or $11$ or $13$. The difference of the last three digits and the remainder digits is divisible by $7$ or $11$ or $13$.

Example: $1864 = 1800 + 64$. By property (4), $1864$ is divisible by $40$ and $5|40$.

Example: $2059282, 1059$ is divisible by $7$ (or $11$ or $13$). The difference of the last three digits and the remainder digits is divisible by $7$ (or $11$ or $13$). The difference is $22$ which is divisible by $11$.

Example 1. Suppose that $x1993y$ is divisible by $45$. Find $x$ and $y$.

Solution. Since $45 = 5 \times 9$, we have $5|x1993y$ and $9|x1993y$. By property (2), $y$ is either $0$ or $5$.

(i) When $y = 0$, since $9|x1993y$, by property (3), $9|(x + 1 + 9 + 9 + 3 + 0)$, thus $x = 5$.

(ii) When $y = 5$, since $9|x1993y$, by property (3), $9|(x + 1 + 9 + 9 + 3 + 5)$, $x = 9$.

Example 2. Suppose that $11|1e2a3a4a5a$. Find $a$.

Solution. By property (7), $11|(5a - (1 + 2 + 3 + 4 + 5))$. That is, $11|(5a - 15)$. Since $5a - 15 = 5(a - 3)$, $11|a - 3$. Since $a$ is between $0$ and $9$, $a = 3$.

Example 3. Suppose that $26|x1991y$. Find $x$ and $y$.

Solution. Since $26 = 2 \times 13$ and $(2, 13) = 1$, $2|x1991y$ and $13|x1991y$.

Since $2|x1991y$, $y$ is $0, 2, 4, 6$, or $8$. We will look at each case.

(1) When $y = 0$, since $13|x19910$, by property 6, we have $13|(x19 - 910)$. Since $13|910, 13|x19$. Note that $x19 = (7 \times 13 + 9)x + 19 = 7 \times 13x + 9x + 13 + 6$.

By property (1), $13|9x + 6$. By guessing and checking, $x = 8$.

(2) When $y = 2$, since $13|x19912$, by property 6, we have $13|(x19 - (910 + 2))$. Since $13|910, 13|(x19 - 2)$. Note that $x19 - 2 = (7 \times 13 + 9)x + 19 - 2 = 7 \times 13x + 9x + 13 + 4$.

By property (1), $13|9x + 4$. By guessing checking, $x = 1$.

Similarly,

(3) When $y = 4$, $x = 7$. 
(4) When $y = 6$, no solution.
(5) When $y = 8$, $x = 6$. 

(1) Given that $72|x931y$. Find $x$ and $y$.

(2) Given that $154xy$ is divisible by 8 and 9. Find $x + y$.

(3) Suppose that $32x5y$ is divisible by 2, 3, 5. Find $x$ and $y$.

(4) Is the number of 1993 digits, $123456789123456789 \cdots$, divisible by 3.

(5) Show that $abcabc$ is divisible by 7, 11, and 13.