Math 113 (Calculus II)  
Midterm Exam 1  
Solutions

Instructions:

• Work on scratch paper will not be graded.

• For questions 6 to 11, show all your work in the space provided. Full credit will be given only if the necessary work is shown justifying your answer. Please write neatly.

• Should you have need for more space than is allotted to answer a question, use the back of the page the problem is on and indicate this fact.

• Simplify your answers. Expressions such as \( \ln(1) \), \( e^0 \), \( \sin(\pi/2) \), etc. must be simplified for full credit.

• Calculators are not allowed.

For Instructor use only.

<table>
<thead>
<tr>
<th>#</th>
<th>Possible</th>
<th>Earned</th>
<th>#</th>
<th>Possible</th>
<th>Earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC</td>
<td>15</td>
<td></td>
<td>9c</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6a</td>
<td>15</td>
<td></td>
<td>9d</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6d</td>
<td>10</td>
<td></td>
<td>9e</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td></td>
<td>9f</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td></td>
<td>10</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>9a</td>
<td>5</td>
<td></td>
<td>11a</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>9b</td>
<td>5</td>
<td></td>
<td>11b</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Sub</td>
<td>65</td>
<td></td>
<td>Sub</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
Multiple Choice. Fill in the answer to each problem on your computer-scored answer sheet. Make sure your name, section and instructor are on that sheet.

1. Find the volume of the solid obtained by rotating the region bounded by the curves \( y = \sec(x) \), \( y = 0 \), \( x = 0 \), \( x = \frac{\pi}{4} \) about the \( x \)-axis.

   a) 1 
   b) 2 
   c) 3 
   d) \( \pi \) 
   e) \( \frac{\pi}{2} \) 
   f) \( \frac{\pi}{3} \) 
   g) None of the above

   ANSWER: D

2. Find the average value of the function \( f(x) = \sqrt[3]{x} \) on the interval \([1, 8]\).

   a) 12 
   b) \( \frac{12}{7} \) 
   c) \( \frac{45}{4} \) 
   d) \( \frac{3}{2} \) 
   e) \( \frac{3}{14} \) 
   f) \( \frac{45}{28} \) 
   g) None of the above

   ANSWER: F

3. If \( f(1) = 2, f(4) = 7, f'(1) = 5, f'(4) = 3 \), and \( f''(x) \) is continuous, what is \( \int_1^4 xf''(x) \, dx \)?

   a) 9 
   b) \( -2 \) 
   c) 12 
   d) \( -1 \) 
   e) 2 
   f) None of the above.

   ANSWER: E

4. What is \( \int_0^{\frac{\pi}{4}} \sin^2(2\theta) \, d\theta \)?

   a) 1 
   b) 0 
   c) \( \frac{1}{2} \) 
   d) \( \frac{\pi}{8} \) 
   e) \( \frac{\pi - 2}{8} \) 
   f) None of the above

   ANSWER: D
5. What is the best form for the partial fraction decomposition of \( \frac{2x + 1}{(x + 1)^3(x^2 + 4)^2} \)?

a) \( \frac{A}{(x + 1)^3} + \frac{Bx + C}{(x^2 + 4)^2} \)

b) \( \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3} + \frac{Dx + E}{x^2 + 4} + \frac{Fx + G}{(x^2 + 4)^2} \)

c) \( \frac{A}{x + 1} + \frac{B}{(x + 1)^3} + \frac{Cx + D}{x^2 + 4} \)

d) \( \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3} + \frac{D}{x^2 + 4} + \frac{E}{(x^2 + 4)^2} \)

e) \( \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3} + \frac{Dx + E}{x^2 + 4} + \frac{Fx + G}{(x + 1)^3(x^2 + 4)^2} \)

f) None of the above

ANSWER: B
6. Consider the region between the curves \( y = 5x \) and \( y = x^2 \) in the first quadrant.

(a) (15 points) Set up an integral for the area of the region bounded by the curves. DO NOT EVALUATE.
ANSWER: \[ \int_0^5 5x - x^2 \, dx \text{ or } \int_0^{25} \sqrt{y} - \frac{y}{5} \, dy \]

(b) Set up an integral for the volume obtained when the region is rotated about the \( x \)-axis. DO NOT EVALUATE.
ANSWER: \[ \int_0^5 \pi (5x^2 - x^4) \, dx \text{ or } \int_0^{25} 2\pi y (\sqrt{y} - \frac{y}{5}) \, dy \]

(c) Set up an integral for the volume obtained when the region is rotated about the \( y \)-axis. DO NOT EVALUATE.
ANSWER: \[ \int_0^{25} \pi (y - \frac{y^2}{25}) \, dy \text{ or } \int_0^5 2\pi x (5x - x^2) \, dx \]

(d) (10 points) Set up an integral for the volume obtained when the region is rotated about the line \( y = -2 \). DO NOT EVALUATE.
ANSWER: \[ \int_0^5 \pi [(5x + 2)^2 - (x^2 + 2)^2] \, dx \text{ or } \int_0^{25} 2\pi (y + 2)(\sqrt{y} - \frac{y}{5}) \, dy \]

(e) Set up an integral for the volume obtained when the region is rotated about the line \( x = -3 \). DO NOT EVALUATE.
ANSWER: \[ \int_0^{25} \pi [ \sqrt{y} + 3 - (\frac{y}{5} + 3)^2 ] \, dy \text{ or } \int_0^5 2\pi (x + 3)(5x - x^2) \, dx \]
7. (5 points) A 12-ft chain weighs 36 lbs and hangs over the edge of a 20 ft high building. How much work is done in pulling the chain to the top of the building?

**ANSWER:**
The chain weighs \( \frac{36}{12} \) lbs per foot, or 3 lbs per foot. The work to raise one slice of length \( dz \) is force times distance, so it is \( W_{slice} = 3z \, dz \). Thus the total work is
\[
W = \int_{0}^{12} 3z \, dz = \frac{3}{2} z^2 \bigg|_{0}^{12} = \frac{3}{2} (12)^2 = 3(72) = 216 \text{ ft-lb}.
\]

8. (10 points) The base of a solid is a circular disk with radius 3. Find the volume of the solid if parallel cross-sections perpendicular to the base are isosceles right triangles with one of the two equal sides lying along the base.

**Answer:**
The volume of a slice is the area of the triangle times the width of the triangle, so it is \( V_{slice} = \frac{1}{2} (2x)^2 \, dy = 2x^2 \, dy \). But we need \( x \) in terms of \( y \), so we use the equation of the circle to get \( x = \sqrt{9 - y^2} \). Thus our volume is:
\[
V = 2 \int_{0}^{3} 2(\sqrt{9 - y^2})^2 \, dy
= 4 \int_{0}^{3} 9 - y^2 \, dy
= 4(9y - \frac{1}{3}y^3)_{0}^{3}
= 4(27 - \frac{1}{3}27)
= 4(27 - 9)
= 72
\]
9. Integrate the following and show all of your work:

(a) (5 points) \( \int \sin^6 x \cos^3 x \, dx \)

Answer:

\[
\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x (1 - \sin^2 x) \cos x \, dx \text{ let } u = \sin x, \text{ then } du = \cos x \, dx \\
= \int u^6(1 - u^2) \, du \\
= \int u^6 - u^8 \, du \\
= \frac{1}{7}u^7 - \frac{1}{9}u^9 + C \\
= \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C
\]

(b) (5 points) \( \int t^5 \ln t \, dt \)

Answer:

Integration by parts: \( u = \ln t \) \quad \( dv = t^5 \, dt \)
\[
\begin{align*}
du &= \frac{1}{t} \, dt \\
v &= \frac{1}{6}t^6
\end{align*}
\]

\[
\int_2^6 t^5 \ln t \, dt = \frac{1}{6}t^6 \ln t - \int_2^6 \frac{1}{6}t^5 \, dt \\
&= \left[ \frac{1}{6}t^6 \ln t - \frac{1}{36}t^6 \right]_2 \\
&= \frac{1}{6}(6)^6 \ln 6 - \frac{1}{36}(6)^6 - \frac{1}{6}(2)^6 \ln 2 + \frac{1}{36}(2)^6
\]
(c) (5 points) \( \int \frac{\ln(\ln x)}{x \ln x} \, dx \)

Answer:
Let \( u = \ln x \). Then \( du = \frac{1}{x} \, dx \). Thus we have

\[
\int \frac{\ln \ln x}{x \ln x} \, dx = \int \frac{\ln u}{u} \, du \quad \text{Let } v = \ln u, \text{ then } dv = \frac{1}{u} \, du
\]

\[
= \int v \, dv
\]

\[
= \frac{1}{2} v^2 + C
\]

\[
= \frac{1}{2} (\ln u)^2 + C
\]

\[
= \frac{1}{2} (\ln \ln x)^2 + C
\]

(d) (5 points) \( \int x \sin 7x \, dx \)

Answer:
Integration by parts:

\[
u = x \quad dv = \sin 7x \, dx
\]

\[
du = dx \quad v = -\frac{1}{7} \cos 7x
\]

\[
\int x \sin 7x \, dx = -\frac{x}{7} \cos 7x - \int -\frac{1}{7} \cos 7x \, dx
\]

\[
= -\frac{x}{7} \cos 7x + \frac{1}{49} \sin 7x + C
\]
(e) (5 points) \( \int x^3 \sqrt{x^2 + 1} \, dx \)

Answer: Let \( x = \tan \theta \), then \( dx = \sec^2 \theta \, d\theta \).

\[
\int x^3 \sqrt{x^2 + 1} \, dx = \int \tan^3 \theta (\tan^2 \theta + 1)^{1/2} \sec^2 \theta \, d\theta \\
= \int \tan^3 \theta \sec^3 \theta \, d\theta \\
= \int (\tan^2 \theta \sec^2 \theta) \tan \theta \sec \theta \, d\theta \\
= \int (\sec^2 \theta + 1) \sec \theta \tan \theta \, d\theta \text{ Let } u = \sec \theta, \, du = \sec \theta \tan \theta \, d\theta \\
= \int (u^2 + 1) u \, du \\
= \int u^4 + u^2 \, du \\
= \frac{1}{5} u^5 + \frac{1}{3} u^3 + C \\
= \frac{1}{5} \sec^5 \theta + \frac{1}{3} \sec^3 \theta + C \\
= \frac{1}{5} (x^2 + 1)^{5/2} + \frac{1}{3} (x^2 + 1)^{3/2} + C
\]

(f) (5 points) \( \int e^{2\theta} \cos 4\theta \, d\theta \)

Answer:

Integration by parts: \( u = e^{2\theta} \quad dv = \cos 4\theta \, d\theta \)

\[ du = 2e^{2\theta} \, d\theta \quad v = \frac{1}{4} \sin 4\theta \]

\[
\int e^{2\theta} \cos 4\theta \, d\theta = \frac{1}{4} e^{2\theta} \sin 4\theta - \frac{1}{2} \int e^{2\theta} \sin 4\theta \, d\theta
\]

Integration by parts again: \( u = e^{2\theta} \quad dv = \sin 4\theta \, d\theta \)

\[ du = 2e^{2\theta} \, d\theta \quad v = -\frac{1}{4} \cos 4\theta \]

\[
\int e^{2\theta} \cos 4\theta \, d\theta = \frac{1}{4} e^{2\theta} \sin 4\theta - \frac{1}{2} \left[ -\frac{1}{4} e^{2\theta} \cos 4\theta + \int \frac{1}{2} e^{2\theta} \cos 4\theta \, d\theta \right]
\]

\[
\frac{5}{4} \int e^{2\theta} \cos 4\theta \, d\theta = \frac{1}{4} e^{2\theta} \sin 4\theta + \frac{1}{8} e^{2\theta} \cos 4\theta + C
\]

\[
\int e^{2\theta} \cos 4\theta \, d\theta = \frac{4}{5} \left[ \frac{1}{4} e^{2\theta} \sin 4\theta + \frac{1}{8} e^{2\theta} \cos 4\theta \right] + C
\]

\[
= \frac{1}{5} e^{2\theta} \sin 4\theta + \frac{1}{10} e^{2\theta} \cos 4\theta + C
\]
10. (5 points) A force of 12 lb is required to hold a spring stretched 3 in. beyond its natural length. How much work is done in stretching it from its natural length to 4 in. beyond its natural length?

ANSWER: Recall that we need our units in feet, so 3 in = $\frac{1}{4}$ foot.

\[ F = kx \]
\[ 12 = \frac{k}{4} \]
\[ k = 48 \]

Thus we have that the work to stretch the spring to 4 in = $\frac{1}{3}$ feet is:

\[ W = \int_0^{\frac{1}{3}} 48x \, dx \]
\[ = 24x^2 \bigg|_0^{\frac{1}{3}} \]
\[ = 24\left(\frac{1}{9}\right) \]
\[ = \frac{8}{3} \text{ ft-lb} \]

11. Integrate the following:

(a) (5 points) \( \int x\sqrt{1-x^4} \, dx \)

Answer:

Let \( x^2 = \sin \theta \). Then \( 2x \, dx = \cos \theta \, d\theta \).

\[ \int x\sqrt{1-x^4} \, dx = \int \frac{1}{2} \sqrt{1-\sin^2 \theta} \cos \theta \, d\theta \]
\[ = \frac{1}{2} \int \cos^2 \theta \, d\theta \]
\[ = \frac{1}{2} \int \frac{1}{2} (1 + \cos 2\theta) \, d\theta \]
\[ = \frac{1}{4} \int 1 + \cos 2\theta \, d\theta \]
\[ = \frac{1}{4} (\theta + \frac{1}{2} \sin 2\theta) + C \]
\[ = \frac{1}{4} [\theta + \frac{1}{2} (2 \sin \theta \cos \theta)] + C \]
\[ = \frac{1}{4} (\sin^{-1} x^2 + x^2 \sqrt{1-x^4}) + C \]
(b) (5 points) $\int_{2}^{3} \frac{2x + 3}{(x - 1)(x + 4)} \, dx$

ANSWER:

First we need to find the partial fraction decomposition of the integrand.

$$\frac{2x + 3}{(x - 1)(x + 4)} = \frac{A}{x - 1} + \frac{B}{x + 4}$$

$$2x + 3 = A(x + 4) + B(x - 1)$$

At $x = -4$ we have $2(-4) + 3 = -5B$. Thus $B = 1$. At $x = 1$ we have $2 + 3 = 5A$, so $A = 1$ also. Thus our integral becomes:

$$\int_{2}^{3} \frac{2x + 3}{(x - 1)(x + 4)} \, dx = \int_{2}^{3} \frac{1}{x - 1} + \frac{1}{x + 4} \, dx$$

$$= \ln |x - 1| + \ln |x + 4||_{2}^{3}$$

$$= \ln 2 + \ln 7 - (\ln 1 + \ln 6)$$

$$= \ln 2 + \ln 7 - \ln 6$$

$$= \ln \frac{14}{6}$$

$$= \ln \frac{7}{3}$$

END OF EXAM